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Reviewed work(s):
Source: *Monographs of the Society for Research in Child Development*, Vol. 30, No. 1, Mathematical Learning: Report of a Conference Sponsored by the Committee on Intellective Processes Research of the Social Science Research Council (1965), pp. 50-59
Published by: Blackwell Publishing on behalf of the Society for Research in Child Development
Stable URL: http://www.jstor.org/stable/1165708
Accessed: 25/01/2012 06:57

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Representation and Mathematics Learning

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The central concern of the present study is the psychological processes involved in the learning of mathematics by children who, in Piaget's sense, are in the stage of "concrete operations" and are not, presumably, yet able to deal readily with formal propositions. Better to understand how mathematics learning of a highly symbolized type might occur, we worked with a small number of children, observing them in minute detail to determine the steps involved in grasping mathematical ideas. Such an approach is, in our opinion, most pressingly needed at this stage of development of new mathematical instruction. It is closely akin to the detailed study of the naturalist and clinician. Perhaps such study can serve to aid more large-scale psychometric testing or, indeed, to elucidate the nature of instruction. It would be disingenuous to say that we (or any naturalist, for that matter) worked without due regard to some theory. Our theoretical predilections were, we should say, far clearer when we finished than when we started. They will also be plain to the reader as our account progresses.

The observations to be reported were made on four eight-year-old children, two boys and two girls, who were given an hour of daily instruction in mathematics four times a week for six weeks. The children were in the IQ range of 120–130 and were enrolled in the third grade of a private school that emphasized instruction designed to foster independent problem-solving. They were from middle-class professional homes. The "teacher" of the class was a well known research mathematician (Z. P. Dienes); his assistant was a professor of psychology at Harvard who has worked long and hard on human thought processes.

Each child worked at a corner table in a generously sized room. Next to each child sat a tutor-observer trained in psychology and with sufficient background in college mathematics to understand the underlying mathematics being taught. In the middle of the room was a large table with a

We are grateful to Dr. Z. P. Dienes, Mr. Samuel Anderson, Miss Eleanor Duckworth, and Miss Joan Rigney for their help in designing and carrying out this study. Dr. Dienes, particularly, formed our thinking about the mode of presenting the mathematical materials.

supply of the blocks and balance beams and cups and beans and chalk that served as instructional aids. In the course of the six weeks, the children were given instruction in factoring, the distributive and commutative properties of addition and multiplication, and, finally, in quadratic functions.

Each child had available a series of graded problem cards to go through at his own pace. The cards gave directions for different kinds of exercises, using the materials described above. The instructor and his assistant circulated from table to table, helping as needed, and each observer-tutor similarly assisted as needed. The problem sequences were designed to provide, first, an appreciation of mathematical ideas through concrete constructions using materials of various kinds for these constructions. From such constructions, the child was encouraged to form perceptual images of the mathematical idea in terms of the forms that had been constructed. The child was then further encouraged to develop or adopt a notation to describe his construction. After such a cycle, a child moved on to the construction of a further embodiment of the idea on which he was working, one that was mathematically isomorphic with what he had learned although expressed in different materials and with altered appearance. When such a new topic was introduced, the children were given a chance to discover its connection with what had gone before and were shown how to extend the notational system used before. Careful minute-by-minute records were kept of the proceedings, along with photographs of the children's constructions.

In no sense can the children, the teachers, the classroom, or the mathematics be said to be typical of what normally occurs in third grade. Four children rarely have six teachers nor do eight-year-olds ordinarily get into quadratic functions. But our concern is with the processes involved in mathematical learning and not with typicality. We would be foolish to claim that the achievements of the children were typical. But it seems quite reasonable to suppose that the thought processes going on in the children were quite ordinary among eight-year-old human beings.

As we have noted, the instruction emphasized concrete construction and embodiment of mathematical concepts. It could have been more axiomatic, less dependent upon visual intuition of forms. It is highly unlikely that there is one optimum procedure for teaching or learning mathematics. The observations obviously reflect the approach of the study as well as the nature of mathematical learning.

Four aspects of the learning seem worth special comment: the role of construction, the uses of notation, the place of contrast and variation, and the character of "insight."

THE ROLE OF CONSTRUCTION

In mathematical factoring, to start with an example, the concept of prime numbers appears to be more readily grasped when the child, through construction, discovers that certain handfuls of beans cannot be laid out

in completed multiple rows and columns. Such quantities have either to be laid out in a single file or in an incomplete row-column design in which there is always one extra or one too few to fill the pattern. These patterns, the child learns, happen to be called "prime" or they could be called "unarrangeable." It is easy for the child to go from this step to the recognition that a multiplication table, so called, is a record sheet of quantities in *completed* multiple rows and columns. Here is factoring, multiplication, and primes in a construction that can also be visualized. Take the matter of factoring in another physical embodiment: a balance beam with hooks placed equidistant from a central fulcrum is the construction vehicle this time (Fig. 1). Contrast this with factoring as the usual computational exercise—as in the problem, "what are the factors of 18?" Conventionally, the child parrots the

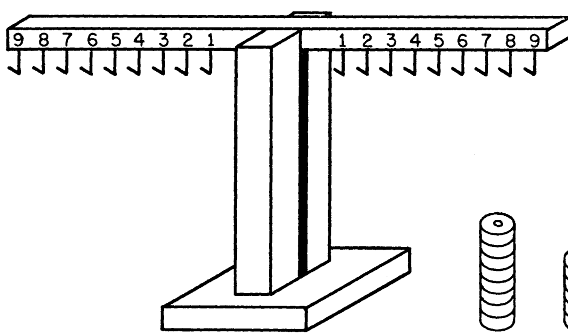


FIGURE 1.—Balance beam and rings used on quadratic construction.

correct set of factors with the usual uncertainty about whether 9 and 2 are different from 2 and 9, or 6 and 3 from 3 and 6. On the balance beam, we place 2 rings on hook 9; the child is encouraged to find and write down every combination of rings on hooks on the opposite side that will balance it. It is a beautiful discovery that 2 rings on hook 9 balances 9 rings on hook 2—and an introduction to the idea of commutativity. Note again that the construction produces a basis for imagery. And before long some startlingly abstract principles couched in elegant terms emerge: "You can exchange rings for hooks if you want." Factors are now events. When notation is applied now, there is a referent.

Note that constructions can be "unconstructed and reconstructed" even when the child does not yet have a ready symbol system for doing so abstractly. In short, construction, unconstruction, and reconstruction provides reversibility in *overt* operations until the child, in Piaget's sense, can internalize such operations in symbolized form.

Now consider quadratic functions. Each child was provided with building materials. These were large flat squares made of wood whose dimensions were unspecified and described simply as "unknown or x long and x wide" (Fig. 2). There were also a large number of strips of wood that were

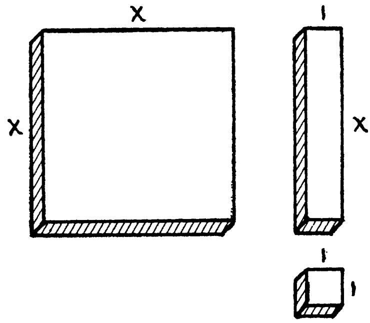


FIGURE 2.—Three components for quadratic constructions.

as long as the sides of the square and described arbitrarily as having a width of "1" or simply "1 by *x*." And there was a supply of little squares with sides equal to the width "1" of the strips, thus "1 by 1." The reader should be warned that the presentation of these materials is not as simple as all that. To begin with, it is necessary to convince the children that we really do not know and do not *care* what is the metric size of the big squares, that rulers are of no interest. A certain humor helps establish in the pupils a proper contempt for measuring in this context, and the snob appeal of simply calling an unknown by the name "*x*" is very great. From there on, the children readily discover for themselves that the long strips are *x* long—by correspondence. They take on faith (as they should) that the narrow dimension is "1," but that they grasp its arbitrariness is clear from one child's declaration of the number of such "1" lengths that make an *x*. As for "1 by 1" little squares, that too is established by simple correspondence with the narrow dimension of the "1 by *x*" strips. It is horseback method but quite good mathematics.

The child is asked whether he can make a square bigger than the *x* by *x* square, using the materials at hand. He very quickly builds squares with

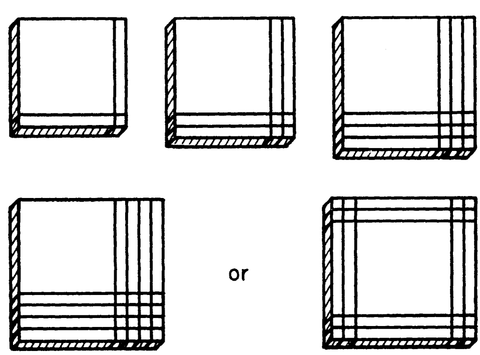


FIGURE 3.—Squares of ever increasing size constructed with components.

designs like those in Figure 3. We ask him to record how much wood is needed for each larger square and how long and wide each square is.

THE USE OF NOTATION

He describes one of his constructed squares; very concretely the pieces are counted out: "an x-square, two x-strips, and a one square" or, "an x-square, four x-strips, and four ones," or "an x-square, six x-strips and nine ones," etc. We help him with language and show him a way to write it down. The big square is an "$x^{\square}$", the long strips are "1 x" or simply "x," and the little squares are "one squares" or "one by one" or better still simply "1." And the expression "and" can be shortened to "+." And so he can write out the recipe for a constructed square as "$x^{\square} + 4x + 4$." At this stage, these are merely names put together in little sentences. How wide and long is the square in question? This the child can readily measure off —an x and 2 or $x + 2$—and so the whole thing is $(x + 2)^{\square}$. Brackets are not so easily grasped. And so the child is able to put down his first equality: $(x + 2)^{\square} = x^{\square} + 4x + 4$. Virtually everything has a referent that can be pointed to with a finger. He has a notational system into which he can translate the image he has constructed.

Now we go on to making bigger squares, and each square the child makes he describes in terms of what wood went into it and how wide and how long it is. It takes some ruled sheets to get the child to keep his record so that he can go back and inspect it for what it may reveal, and he is encouraged to go back and look at the record and at the constructions they stand for.

Imagine now a list such as the following, again a product of the child's own construction:

$x \quad + 2x + \quad 1$ is $x + 1$ by $x + 1$
$x \quad + 4x + \quad 4$ is $x + 2$ by $x + 2$
$x \quad + 6x + \quad 9$ is $x + 3$ by $x + 3$
$x \quad + 8x + 16$ is $x + 4$ by $x + 4$

It is almost impossible for him not to make some discoveries about the numbers: that the x values go up 2, 4, 6, 8 . . . and the unit values go up 1, 4, 9, 16 . . . and the dimensions increase by additions to x of 1, 2, 3, 4 . . . . The syntactical insights about regularity in notation are matched by perceptual-manipulative insights about the material referents.

After a while, some new manipulations occur that provide the child with a further basis for notational progress. He takes the square, $(x + 2)^2$, and reconstructs it in a new way (Fig. 4). One may ask whether this is constructive manipulation or whether it is proper factoring. But the child is learning that the same amount of wood can build quite strikingly different patterns and remain the same amount of wood—although it also has a different notational expression. Where does the language begin and the

manipulation of materials stop? The interplay is continuous. We shall return to this same example in a later section.

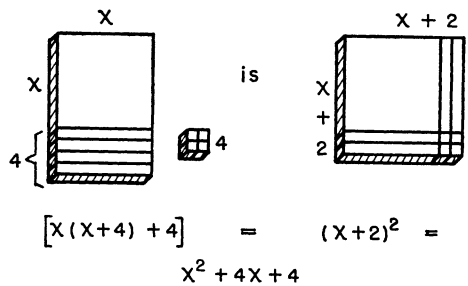


FIGURE 4.—Syntactic exercise supported by construction.

But the problem now is how to "detach" the notation that the child has learned from the concrete, visible, manipulable embodiment to which it refers—the wood. For if the child is to deal with mathematical properties he will have to deal with symbols *per se*, else he will be limited to the narrow and rather trivial range of symbolism that can be given direct (and only partial) visual embodiment. Concepts such as x^2 and x^3 may be given a visualizable referent, but what of x^n?

Why do children wean themselves from the perceptual embodiment to the symbolic notation? Perhaps it is partly explained in the nature of variation and contrast.

VARIATION AND CONTRAST

The child is shown the balance beam again and told, "Choose any hook on one side and put the same number of rings on it as the number the hook is away from the middle. Now balance it with rings placed on the other side. Keep a record." Recall that the balance beam is familiar from work on factoring and that the child knows that 2 rings on 9 balances 9 on 2 or m rings on n balances n on m. He is back to construction. Can anything be constructed on the balance beam that is like the squares? With little effort, the following translation is made. Suppose x is 5. Then 5 rings on hook 5 is x^2, five rings on hook 4 is $4x$, and 4 rings on hook 1 is 4: $x^2 + 4x + 4$. How can we find whether this is like a square that is $x + 2$ wide by $x + 2$ long as before? Well, if x is 5, then $x + 2$ is 7, and so 7 rings on hook 7. And nature obliges—the beam balances. One notation works for two strikingly different constructions and perceptual events. Notation, with its broader equivalency, is clearly more economical than reference to embodiments. There is little resistance to using this more convenient language. And now construction can begin—commutative and distributive

properties of equations can be explored: $x(x + 4) + 4 = x^2 + 4x + 4$ or $x + 4$ rings on hook x and 4 rings on hook 1 will balance. The child, if he wishes, can also go back to the wood and find that the same materials can make the design in Figure 4.

Contrast is the vehicle by which the obvious that is too obvious to be appreciated can be made noticeable again. The discovery of an eight-year-old girl illustrates the matter. "Yes, 4×6 equals 6×4 in numbers, like in one way six eskimos in four igloos is the same as four in six igloos. But a venetian blind *isn't* the same as a blind Venetian." By recognizing the noncommutative property of ordinary language, the commutative property of a mathematical language can be partly grasped. But it is still only a partial insight into commutativity and noncommutativity. Had we wished to develop the distinction more deeply we might have proceeded concretely to a contrast between sets of operations that can be carried out in any sequence—like the order of eating courses at a dinner or of going to different movies—and operations that have a noncommutative order—like putting on shoes and socks—where one must precede the other. Then the child could be taken from there to a more general idea of commutative and noncommutative cases and ways of dealing with a notation, perhaps by identical sets and ordered identical sets.

INSIGHT AND DEVELOPMENT

What was so striking in the performance of the children was their *initial* inability to represent things to themselves in a way that transcended immediate perceptual grasp. The achievement of more comprehensive insight requires, we think, the building of a mediating representational structure that transcends such immediate imagery, that renders a *sequence* of acts and image unitary and simultaneous. The children always began by constructing an embodiment of some concept, building a concrete form of operational definition. The fruit of the construction was an image and some operations that "stood for" the concept. From there on, the task was to provide means of representation that were free of particular manipulations and specific images. Only symbolic operations provide the means of representing an idea in this way. But consider this matter for a moment.

We have already commented upon the fact that by giving the child multiple embodiments of the same general idea expressed in a common notation we lead him to "empty" the concept of specific sensory properties until he is able to grasp its abstract properties. But surely this is not the best way of describing the child's increasing development of insight. The growth of such abstractions is important. But what struck us about the children, as we observed them, is that they had not only understood the abstractions they had learned but also had a store of concrete images that served to exemplify the abstractions. When they searched for a way to deal with new problems, the task was usually carried out not simply by abstract

means but also by "matching up" images. An example will help here. In going from the wood-blocks embodiment of the quadratic to the balance-beam embodiment, it was interesting that the children "equated" *concrete* features of one with *concrete* features of another. One side of the balance beam "stood for" the amount of wood, the other side for the sides of the square. These were important concrete props on which they leaned. We have been told by research mathematicians that the same use of props—heuristics—holds for them, that they have preferred ways of imagining certain problems while other problems are handled silently or in terms of an imagery of the symbolism on a page.

We reached the tentative conclusion that it was probably necessary for a child learning mathematics not only to have as firm a sense of the abstraction underlying what he was working on but, also, a good stock of visual images for embodying them. For without the latter, it is difficult to track correspondences and to check what one is doing symbolically. Here an example will help again. We had occasion, again with the help of Dr. Dienes, of teaching a group of 10 nine-year-olds the elements of group theory. To embody the idea of a mathematical group initially, we gave them the example of a four-group made up of the following four maneuvers (a book was the vehicle, a book with an arrow up the middle of its front cover): rotating the book a quarter turn to the left, rotating it a quarter turn to the right, rotating it a half-turn (without regard to direction of rotation), and letting it stay in the position it was in. They were quick to grasp the important property of such a mathematical group: that any sequence of maneuvers made could be reproduced from the starting position by a single move. This is not the usual way in which this property is described mathematically, but it served well for the children. We contrasted this elegant property with a series of our moves that did *not* constitute a mathematical group—indeed, they provided the counter-example themselves by proposing the one-third turn left, one-third turn right, half-turn either way, and stay. It was soon apparent that it did not work. We set the children the task of making games of four maneuvers, six maneuvers, etc., that had the property of a "closed" game, as we called it. They were, of course, highly ingenious. But what soon became apparent was that they needed some aid in imagery—in this case an imagery notation—that would allow them to keep track and then to discover whether some new game was an isomorph of one they had already developed. The prop in this case was, of course, the matrix, listing the moves possible across the top and then listing them down the side, thus making it easily possible to check whether each combination of pairs of moves could be reproduced by a single move. The matrix in this case is a crutch or heuristic and as such has nothing to do with the abstraction of the mathematical group, yet it was enormously useful to them not only for keeping track but also for comparing one group with another for correspondence. Thus the matrix with which they started had the property of

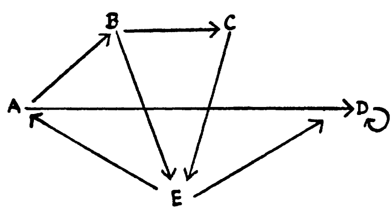
	s	*a*	*b*	*c*
s	*s*	*a*	*b*	*c*
a	*a*	*c*	*s*	*b*
b	*b*	*s*	*c*	*a*
c	*c*	*b*	*a*	*s*.

Are there any four groups with a different structure? It is extremely difficult to deal with such a question without the aid of this housekeeping matrix as a vehicle for spotting correspondence.

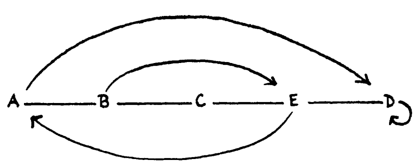
A still better example is provided by a colleague, pointing to the role of imagery in dealing with certain formal properties.[1] Suppose we specify the permissible moves in a finite state structure consisting of the states *A*, *B*, *C*, *D*, *E*. One may list the permissible transitions between states as follows:

AB
AD
BC
BE
CE
DD
ED
EA

Suppose we now ask of someone who has this set of rules for moving among the five states what is the shortest path from *A* to *E* that moves through C. Even with the ordered information in the list, it takes a moment to figure it out. How much easier the task becomes when one produces an image to carry the information, such as,



or better, the following:



Much of mathematics is carried out with just such "less-than-rigorous" technique, and it is likely as important as abstraction in the actual *doing* of mathematical problems. One can use highly concrete embodiments to serve

[1] We are grateful for this example to Dr. Richard Hays.

such uses. The building blocks used in teaching quadratic functions can serve as a "source image" for checking and rethinking just as readily as the diagraming of finite state structures noted directly above.

In sum, then, while the development of insight into mathematics in our group of children depended upon their development of "example-free" abstractions, this did not lead them to give up their imagery. Quite to the contrary, we had the impression that their enriched imagery was very useful to them in dealing with new problems.

We would suggest that learning mathematics may be viewed as a microcosm of intellectual development. It begins with instrumental activity, a kind of definition of things by doing. Such operations become represented and summarized in the form of particular images. Finally, and with the help of a symbolic notation that remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with. But while, once abstraction is achieved, the learner becomes free in a certain measure of the surface appearance of things, he nonetheless continues to rely upon the stock of imagery he has built en route to abstract mastery. It is this stock of imagery that permits him to work at the level of heuristic, through convenient and non-rigorous, means of exploring problems and relating them to problems already mastered.